Vanguard Research Initiative Technical Report: Long-term Care Model

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This document describes an algorithm to compute optimal saving and expenditure policies in a model featured in papers associated with the Vanguard Research Initiative. This model appears in Ameriks, Briggs, Caplin, Shapiro, and Tonetti (2015) and Ameriks, Briggs, Caplin, Shapiro, and Tonetti (2016).

1 Model Environment

1.1 Consumers

Consumers are heterogeneous over wealth ($a \in [0, \infty)$), income age-profile ($y \in \{y_1, y_2, \ldots, y_5\}$), age ($t \in \{55, 56, ..., 108\}$), gender ($g \in \{m, f\}$), health status ($s \in \{0, 1, 2, 3\}$), and health cost ($h \sim H(t, s)$ with support $\Omega_H(t, s)$). Time is discrete and the life-cycle horizon is finite. Consumers start at age t_0 and live to be at most T-1 years old, where in our parameterization t_0 corresponds with age 55 and T corresponds with age 108. Each period individuals choose consumption ($c \in [0, \infty)$), savings ($a' \geq 0$), and whether to use government care ($G \in \{0, 1\}$). The model groups people into five income groups with deterministic age-income profiles.[1] Each individual has a perfectly foreseen deterministic income sequence and receives a risk free rate of return of $(1 + r)$ on savings. The only uncertainty an individual has is over health/death.

1.2 Government

The consumer always has the option to use a means-tested government provided care program. The cost of using government care is that a consumer's wealth is set to zero, while the benefit is that the government provides predetermined levels of expenditure, which depend on the health status of the individual as described below. $G = 1$ if the consumer chooses to use government care and $G = 0$ if the consumer chooses not to use government care.

[1] The model abstracts from labor supply decisions, including retirement. These labor market decisions are taken into account through the exogenous income profiles.

1.3 Health and Death

There are four health states: $s = 0$ represents good health, $s = 1$ represents poor health, $s = 2$ represents the need for long-term care (LTC), and $s = 3$ represents death. The health state evolves according to a Markov process, where the probability matrix, $\pi_g(s'|t, s)$ is gender, age, and health state dependent. h is a stochastic health expenditure that must be paid—essentially a negative wealth shock. Each period the consumer has to pay this health cost, h, where, $h \sim H(t, s)$ and H is the CDF of the health cost random variable with support $\Omega_H(t, s)$.

If a consumer chooses to use government care when he does not need LTC (i.e., when $s = 0, 1$), then the government provides a consumption floor, $c = \omega_G$, that is designed to represent welfare.

A consumer needs LTC if he needs help with the activities of daily living (ADLs), such as bathing, eating, dressing, walking across a room, or getting in or out of bed. Thus, state 2 is interchangeably referred to as the LTC or ADL state. If a consumer needs LTC ($s = 2$), then he must either purchase private long-term care or use government care. Capturing the fact that LTC provision is essential for those in need and private long-term care is expensive, there is a minimum level of expenditure needed to obtain private LTC, i.e., $c \geq \chi$ for those not using government care. In the model, government-provided care is loosely based on the institutions of Medicaid. If a consumer needs LTC and uses government care, the government provides $c = \psi_G$. The value ψ_G parameterizes the consumer's value of public care, since that parameter essentially determines the utility of an individual who needs LTC and chooses to use government care.

In addition to affecting health costs and survival probabilities, health status affects preferences. There is a health-dependent utility function, such that spending when a consumer needs LTC ($s = 2$) is valued differently than spending when a consumer does not need LTC. Utility when in need of LTC associated with expenditure level c is

$$(\theta_{ADL})^{-\gamma} \ \frac{(c + \kappa_{ADL})^{1-\gamma}}{1-\gamma}.$$

Upon death ($s = 3$), the agent receives no income and pays all mandatory health costs. Any remaining wealth is left as a bequest, b, which the consumer values with a warm glow utility function:

$$v(b) = (\theta_{beq})^{-\gamma} \ \frac{(b + \kappa_{beq})^{1-\gamma}}{1-\gamma}.$$

2 The Model

The choice variables are:

$$c \in [0, \infty)$$
$$a \in [0, \infty)$$
$$G \in \{0, 1\}$$

The exogenous state variables are:

$$
\begin{aligned}
& g \in \{\text{male, female}\} \\
& s \in \{0,1,2,3\} \text{ evolves according to } \pi(s'|t,s) \\
& t \in \{55, 56, ..., T\} \\
& y \in \left((y^k(t))_{t=1}^{T}\right)_{k=1}^{5} \\
& h \sim H(t,s) \text{ with support } \Omega_H(t,s)
\end{aligned}
$$

The consumer's problem is

$$
\begin{aligned}
V(a,y,t,s,h,g) = \max_{a',\ c,\ G}\ & \mathbb{I}_{s\neq 3}\ (1-G)\ \left\{U_s(c) + \beta E[V(a',y,t+1,s',h')]\right\} \\
& + \mathbb{I}_{s\neq 3}\ G\ \left\{U_s(\omega_G,\psi_G) + \beta E[V(0,y,t+1,s',h')]\right\} + \mathbb{I}_{s=3}\{v(b)\}
\end{aligned}
$$

$$
\begin{aligned}
& \text{s.t.} \\
& a' = (1-G)[(1+r)a + y(t) - c - h] \geq 0 \\
& c \geq \chi_{ADL} \ \text{ if } \ (G=0 \ \wedge \ s=2) \\
& c = \psi_G \ \text{ if } \ (G=1 \ \wedge \ s=2) \\
& c = \omega_G \ \text{ if } \ (G=1 \ \wedge \ (s=0 \ \vee \ s=1)) \\
& b = \max\{(1+r)a - h'\ ,\ 0\} \\
& U_s(c) = \mathbb{I}_{s\in\{0,1\}}\ \frac{c^{1-\gamma}}{1-\gamma} + \mathbb{I}_{s=2}\ (\theta_{ADL})^{-\gamma}\ \frac{(c+\kappa_{ADL})^{1-\gamma}}{1-\gamma} \\
& v(b) = (\theta_{beq})^{-\gamma}\ \frac{(b+\kappa_{beq})^{1-\gamma}}{1-\gamma}.
\end{aligned}
$$

The G variable is a choice variable that is a function of the consumer's states, i.e., $G(a,y,s,j,h) \in \{0,1\}$. Note that we can suppress the policy function G in favor of a max operator.

$$
V(a,y,t,s,h,g) = \mathbb{I}_{s\neq 3} \max\left\{\max_{a',c} U_s(c) + \beta E[V(a',y,t+1,s',h',g)]\ \ ,\ \ U_s(\omega_G,\psi_G) + \beta E[V(0,y,t+1,s',h',g)]\right\} + \mathbb{I}_{s=3} v(b)
$$

3 Computing Optimal Policies

We solve this finite horizon model using backwards induction on a discretized state space. We first compute optimal consumer policies at age $T-1$, when the continuation value at age T is simple to compute, as it is just the value of leaving a bequest. This yields optimal consumer policies and the value function at each point in the state space at time $T-1$. Knowing these $T-1$ values, we repeat the algorithm for date $T-2$, $T-3$,..., t_0.

We adapt to our problem an extension of the endogenous grid method (EGM) for non-concave problems, building heavily on the algorithm developed in Fella (2014).[2]

3.1 Endogenous grid method

The basic idea behind the EGM is that, rather than solve for the next period asset level given the initial period asset level, it is more efficient to solve for the initial period asset level given the next period asset level. Since at each time we know the expected continuation value (ECV) function from backwards induction, the first order condition (FOC) can be inverted analytically, avoiding the need for computationally costly non-linear equation solving. This speeds up computation time significantly. The solution algorithm proceeds as follows:

3.2 T-1 Problem

- Define value functions
- Apply endogenous grid method
- Check constraints and boundary solutions
- Compare to the value of government care
- Store a matrix that reports the value of $V(a, y, T-1, s, h, g)$ for all $s \in \{0, 1, 2, 3\}$, $h \in \hat{\Omega}_H(T-1, s)$, and $y = (y(t))_{t=t_0}^{T}$. Note that $\hat{\cdot}$ denotes discretized values, e.g., $\hat{\Omega}_H(t, s)$ is the discretized space of health costs for each age and health state.

We start out by solving the $T-1$ problem under the assumption $G = 0$, i.e., the consumer does not use government care. We will later compare this value to the value if the consumer does use government care in a final step. Given that the next period (time T) continuation value is $v(b)$, the $T-1$ bellman equation is

$$J(a, y, T-1, s, h, g) = \max_{c,e,a'} U_s(c) + \beta \int_{\Omega_H(T,s'=3)} v\left(max\left\{(1+r)a' - h', 0\right\}\right) dH(h'|T, s' = 3).$$

Define the expected continuation value, $ECV_{T-1}(a')$, as

$$ECV_{T-1}(a') := \beta \int_{\Omega_H(T,s'=3)} \left[v\left((1+r)a' - h'\right) \times \mathbb{I}_{\{(1+r)a'-h')>0\}}\right] dH(h'|T, s' = 3).$$

Then the FOCs are

$$\frac{d}{dc_{T-1}} U_s(c) = \frac{d}{da'} ECV_{T-1}(a').$$

[2] We have also solved the model by directly calculating the value function using a constrained optimizer. The EGM code runs at least 10 times faster, so it is our chosen algorithm. We verified in a somewhat simpler model that the constrained optimization code and EGM code generate the same optimal policy functions, confirming the accuracy of our modified EGM algorithm.

$$\frac{d}{dc}U_{s=0}(c) = \frac{d}{dc}U_{s=1}(c) = c^{-\gamma} = \big((1+r)a + y(T-1) - a' - h\big)^{-\gamma}$$

$$\frac{d}{dc}U_{s=2}(c) = (\theta_{ADL})^{-\gamma}\,(c+\kappa_{ADL})^{-\gamma} = (\theta_{ADL})^{-\gamma}\,\big((1+r)a + y(T-1) - a' - h + \kappa_{ADL}\big)^{-\gamma}.$$

Thus, if we can solve the above equations for a' for any a then we will have characterized the policy functions for the $T-1$ problem (since given a' and s, the budget constraint provides c).

To solve for the (a, a') pairs that solve the above problem, we proceed with the Endogenous Grid method.

1. First, construct a grid over a', named a_{Final}. For each $a'^{,i} \in a_{Final}$, we solve the FOC for a^i

$$\begin{aligned}
\big((1+r)a^i + y(T-1) - a'^{,i} - h\big)^{-\gamma} &= \frac{d}{da'}ECV_{T-1}(a'^{,i}) \\
a^i &= \frac{\frac{d}{da'}ECV_{T-1}(a'^{,i})^{\frac{1}{-\gamma}} - y(T-1) + a'^{,i} + h}{(1+r)} \\
(\theta_{ADL})^{-\gamma}\big((1+r)a^i + y(T-1) - a'^{,i} - h + \kappa_{ADL}\big)^{-\gamma} &= \frac{d}{da'}ECV_{T-1}(a'^{,i}) \\
a^i &= \frac{\frac{1}{\theta_{ADL}}\left(\frac{d}{da'}ECV_{T-1}(a'^{,i})\right)^{\frac{1}{-\gamma}} - y(T-1) + a'^{,i} + h - \kappa_{ADL}}{(1+r)}
\end{aligned}$$

   Thus, for each $a'^{,i} \in a_{Final}$ we have found a point a^i which satisfies the FOC. Notice that a^i is a function of idiosyncratic states, including s, h, g, and y but we suppress this for notational convenience. However, a^i must be evaluated at each point of the idiosyncratic state space grid (s, h, g, y). These pairs $(a^i, a'^{,i})$ define the policy function as the mapping from the endogenous a grid (hereafter referred to as $a_{Endogenous}$) to $a'^{,i}$.

2. To implement our solution technique, the policy function must be defined over a standardized a grid (call it a_{Start}) to ensure both that the grid is constant in all time periods and that the policy function is defined over the range of the data. Thus, we would like to define $a'(a^j)\ \forall\ a^j \in a_{Start}$. To obtain it, we specify the grid a_{Start}, and for each point $a^j \in a_{start}$ we find $\{(a^i, a'^{,i}), (a^{i+1}, a'^{,i+1})\} \in \{a_{Endogenous}, a_{Final}\}$ such that $a^j \in [a^i, a^{i+1}]$. We then calculate $a'(a^j)$ by interpolating $a'(a^j)$ over $[a'^{,i}, a'^{,i+1}]$. This gives a grid $\{a_{Start}, a'(a_{Start})\}$. After repeating the procedure for each grid point we have computed a set of grids that satisfies the $T-1$ FOC: $\{a_{Start}, a'(a_{Start})\}_{s,h}$, $s \in \{0,1,2\}, h \in \hat{\Omega}_H(T-1, s)$.

3. Remembering that in state 2, there is an LTC expenditure constraint ($e_{LTC} > \chi$), we must check to see if this constraint is satisfied. For each pair in $\{a_{Start}, a'(a_{Start})\}_{s=2}$ such that the budget constraint can be satisfied (i.e, $(1+r)a_{Start} + y(T-1) - h(T-1,2) - \chi > 0$) but $a'(a_{Start}) > (1+r)a_{Start} - 1 - h(T-1,2) - \chi$, we set $a'(a_{Start}) = (1+r)a_{Start} - 1 - h(T-1,2) - \chi$ (or equivalently, $e_{LTC} = \chi$) and proceed.

   For each of the above policy functions and starting grid we calculate the value function. Define $\hat{J}$ as the value of a consumer who does not go on government care evaluated at the optimal element of the a' grid:

$$\begin{aligned}
\hat{J}(a, y, T-1, s, h, g, a') &= U_s(c) + \beta \int_{\Omega_H(T, s'=3)} v\left(max\left\{(1+r)a' - h', 0\right\}\right) dH(h'|T, s'=3) \\
&\approx U_s(c) + \beta \sum_{h'_i \in \hat{\Omega}_H(T, s'=3)} v\left(max\left\{(1+r)a' - h'_i, 0\right\}\right) p(h'_i|T, s'=3)
\end{aligned}$$

where now c is implied by a' and we have approximated the integral numerically. This yields a grid of the value function $\hat{J}(a, y, T-1, s, h, g, a')$ for each non-age idiosyncratic state over all starting asset values $a^j \in a_{Start}$. We now need to check two things. First, we check the boundary condition of whether $\hat{J}(a^j_{Start}, y, T-1, s, h, g, a'(a^j_{Start})) > \hat{J}(a^j_{Start}, y, T-1, s, h, g, 0)$ (i.e., whether an agent is better off consuming all wealth). We need to check this point because of the kink that can occur in the value function due to $b = max\{(1+r)a' - h(T, s'), 0\}$ being bounded below by 0.

4. Finally, we check whether the value of government care,

$$GC(a, y, T-1, s) = U_s(\omega_G, \psi_G)] + \beta v(0),$$

yields a higher utility than the previously calculated value of utility, $\hat{J}(a^j, y, T-1, s, h, g, a'^{,i})$ for each $a^j \in a_{Start}$.

The value function, V, is then the max over the optimal value when $G = 0$ and when $G = 1$:

$$V(a^i, y, T-1, s, h, g) = max\left(\hat{J}(a^j, y, T-1, s, h, g, a'^{,i})\ ,\ GC(a^j, y, T-1, s)\right).$$

Note that if the budget constraint is unable to be satisfied (i.e. $(1+r)a^j + y - h(T-1, s) < 0$), it must be that $V(a^j, y, T-1, s, h, g) = GC(a^j, y, T-1, s)$, since $c^j = e^j_{LTC} = 0$ implies that $V(a^j, y, T-1, s, h, g, a'^{,j}) = -\infty$. For state 3, we define a grid

$$V(a^j, y, T-1, 3, h, g) = v(max\left\{(1+r)a^j + y - h,\ 0\right\})$$

and join it with the grids for states 0, 1, and 2.

Thus, we have calculated a grid of the value function value for an underlying asset grid for the $T-1$ problem for each health state. This will be used to solve the $T-2$ problem as we continue to the backwards induction procedure.

3.3 $T-j$ Problem, $j \geq 2$

The solution to the $T-1$ problem is taken as given and used to solve the $T-2$ problem, which is then used to solve the $T-3$ problem recursively until reaching age t_0. We use the previously calculated $T-j+1$ value function $V(a, y, T-j+1, s, h, g)$ over a grid of $(a, y, s, h, g) \in [a_{min}, a_{max}] \times ((y_k(T-j)))_{k=1}^5 \times \{0, 1, 2, 3\} \times \hat{\Omega}_H(T-j+1, s) \times \{\text{male, female}\}$ to define the expected continuation value (ECV) needed to solve the $T-j$ problem. The algorithm proceeds in a similar manner to that developed for the $T-1$ period problem.

- Define value functions using continuation value from $T-j+1$ problem
- Apply endogenous grid method to non-concave problems
- Check constraints and boundary solutions
- Compare to value of government care
- Store a matrix that reports the value of $V(a, y, T-j, s, h, g)$ for all $s \in \{0, 1, 2, 3\}$, $h \in \hat{\Omega}_H(T-j, s)$, $(y_k(T-j))_{k=1}^5$, $g \in \{\text{male, female}\}$

The $T-j$ expectation of the $T-j+1$ value function, i.e., the expected continuation value ECV_{T-j}, is

$$
\begin{aligned}
ECV_{T-j}(a',s) &= \beta \sum_{s' \in \{0,1,2,3\}} \pi(s'|T-j,s) \int_{\Omega_H(T-j+1,s)} V(a',y,T-j+1,s',h')dH(h'|T-j+1,s') \\
&\approx \beta \sum_{s' \in \{0,1,2,3\}} \pi(s'|T-j,s) \sum_{h'_i \in \hat{\Omega}_H(T-j+1,s')} V(a',y,T-j+1,s',h'_i)\left[H(h'_i|T,s') - H(h'_{i-1}|T,s')\right]
\end{aligned}
$$

and thus the value function at time $T-j$ if the consumer does not go on government care is

$$
J(a,y,T-j,s,h,g) = \max_{c_{T-j},e_{T-j},a'} U_s(c) + ECV_{T-j}(a',s).
$$

The FOC is

$$
\frac{d}{dc}\left[U_s(c)\right] = \frac{d}{da'}\left[ECV(a',s)\right] = \frac{d}{da'}\left[\beta E[V(a',y,j+1,s',h')]\right]
$$

Since $\frac{d}{dc}\left[U_s(c)\right] = \mathbb{I}_{\{s=\{0,1\}\}}c^{-\gamma} + \mathbb{I}_{\{s=2\}}\theta_{ADL}(c+\kappa_{ADL})^{-\gamma}$, if we know $\frac{d}{da'}\left[\beta E[V(a',y,j+1,s',h')]\right]$ then we can substitute in $c=(1+r)a+y-a'-h$ and invert the FOC to obtain initial period asset holdings. Doing this over a grid of final assets a_{Final} will yield an endogenous grid of initial asset holdings, $a_{Endogenous}$ that imply the policy functions $(a^i_{Endogenous}, a^i_{Final})$. This process must be repeated for all $s \in \{0,1,2,3\}$ *and* $h \in \hat{\Omega}_H(T-j,s)$ with a grid being calculated for each point. Analytical expressions of $\frac{d}{da'}\left[\beta E[V(a',y,j+1,s',h')]\right]$ are included in Section 4.

If the expected continuation value $\beta E[V(a',T-j+1,s',h',g)]$ is globally concave, then the FOC will be both necessary and sufficient, and thus the computed endogenous grid, specifically the pairs $(a^i_{Endogenous}, a^i_{Final})$, characterize optimal consumer behavior. If the value function V is not globally concave, then the FOC will only be necessary but not sufficient. The model in this paper features a non-concave value function due to kinks induced by the government care options and expenditure constraints. Thus, the algorithm needs to be adapted to ensure that the optimal policies computed with the EGM correspond to the optimal policies of the model.

To illustrate the complication, suppose that the ECV is not concave. Then the derivative of the ECV might look similar to the plot in Figure 1 taken from Fella (2014).

There is no longer a one-to-one mapping from $\frac{d}{da'}\left[\beta E[V(a',y,T-j+1,s',h',g)]\right]$ to $\frac{d}{dc}\left[U_s(c)\right]$ (where we are adapting our notation) in the regions of $a' \in \{a'_2, ...a'_9\}$. For any a'_i in this set, there exists at least one a'_k such that $a'_i \neq a'_k$ but $\frac{d}{da'}\left[\beta E[V(a'_i,y,T-j+1,s',h',g)]\right] = \frac{d}{da'}\left[\beta E[V(a'_k,y,T-j+1,s',h',g)]\right]$. Thus, the inverted FOC pins down a non-singleton set of admissible next period wealth levels a'_k or a'_i associated with initial period asset level a.

To address this concern, we first partition the a'_{Final} into concave and non-concave regions. In the concave region, the bijective mapping allows for straightforward calculation of optimal (a,a') pairs as discussed above. In the non-concave region, we discard (a, a'_{Final}) that are not a pair of initial assets and optimal savings. That is, we check to see for an initial period asset a, whether the policy a'_k or a'_i would optimize the value function. We do this by finding the a' that is the global $\arg\max$, saving that (a,a') pair to be used in interpolation, and discarding other pairs.

The algorithm is detailed below. Note that here we have suppressed the state dependence notation of

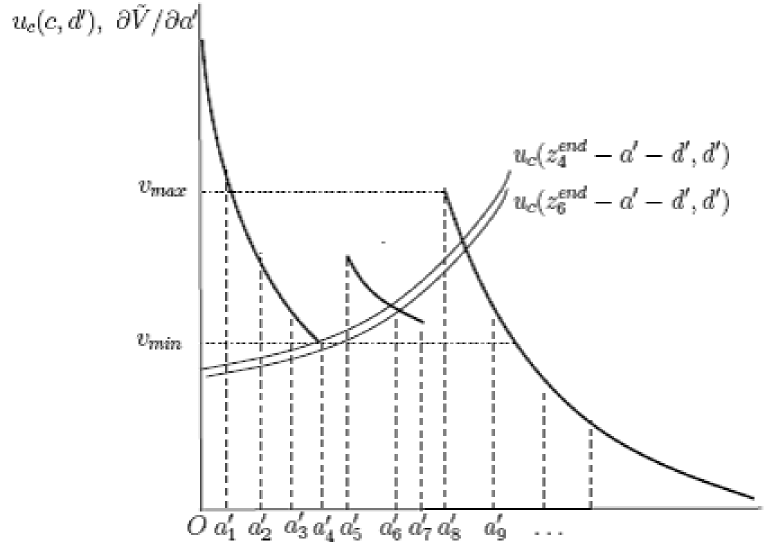

Figure 1: Illustrating the algorithm

Figure 1: Non-concave Value Function FOC (Fella 2014)

the savings policy function and simply expressed it as a'. The below process must be repeated over the entire grid, $(y, s, h, g) \in (y_k(T-j))_{k=1}^{5} \times \{0, 1, 2, 3\} \times \hat{\Omega}_H(T-j, s) \times g \in \{\text{male, female}\}$ to obtain the pairs referenced in step 3.

1. We calculate the lower and upper regions of our asset grid where there is a unique mapping from $\frac{d}{da'}\beta E[V(a', y, T-j+1, s', h')]$ to $\frac{d}{dc}[U_s(c)]$ and thus to initial assets. These regions are determined by the locally concave lower and upper tails of the value function and are labeled G_C.

   (a) In the above figure, the region G_C is characterized by all a_i such that $\frac{d}{da'}[\beta E[V(a_i, y, T-j+1, s', h', g)]] > v_{max}$ or $\frac{d}{da'}\beta E[V(a_i, y, T-j+1, s', h', g)] < v_{min}$, i.e., $a_i < a_2', a_i > a_9'$

   (b) In the G_C region, the standard EGM can be applied without adjustment because the unique mapping implies the FOC is both necessary and sufficient

   We calculate this region by calculating the derivative of ECV_{T-j} for every point $a_i' \in a_{Final}$, yielding $\frac{d}{da'}[ECV_{T-j}(a', s)]$. We then find the regions such that

   (a) $\frac{d}{da'}[ECV_{T-j}(a'^{,i}, s)] < \frac{d}{da'}[ECV_{T-j}(a'^{,j}, s)]$ $\forall j < i$ and

   (b) $\frac{d}{da'}[ECV_{T-j}(a'^{,i}, s)] > \frac{d}{da'}[ECV_{T-j}(a'^{,j}, s)]$ $\forall j > i$.

   We then label the minimum i for which the first condition holds $\underline{i}$, and label the maximum i for which the second condition holds as $\bar{i}$. Finally, G_C contains $a_j \in a_{Final}$ s.t. $j > \underline{i}$ or $j < \bar{i}$. Since the government care option generates a value function that is not concave for low asset values, we will generally only find a concave upper region.

   To complete the partition of a'_{Final}, we store all a' in the non-concave region where there is not a unique mapping between $\frac{d}{da'}\beta E[V(a', y, j+1, s', h')]$ and $\frac{d}{dc}[U_s(c)]$ in G_{NC} .

2. For each $a^i_{Final} \in G_{NC}$, we check pairs $(a^i_{Endogenous}, a^i_{Final})$ to ensure that a^i_{Final} are global arg max for the corresponding $a^i_{Endogenous}$ state. In G_{NC} there may be multiple a'_{Final} mapping to a single $a_{Endogenous}$. Thus, we must find the unique a'_{Final} that is the optimal saving policy if an individual were to start the period with $a_{Endogenous}$ assets. To find the optimal savings policy, we evaluate the objective function for each $a'_i \in G_{NC}$ to find

$$a'_i = \arg\max_{a' \in G_{NC}} U_s((1+r)a_i + y(T-j) - a' - h(j,s)) + ECV_{T-j}(a', s).$$

We store the (a_i, a'_i) pair if this condition holds and discard the suboptimal pairs.

   (a) In the above figure G_{NC} would contain $a_i \in [a'_2, a'_9]$ or equivalently $v_{min} < \frac{d}{da'}\left[\beta E[V(a_i, y, j+1, s')]\right] < v_{max}$.

3. The remaining optimal pairs are $\left\{(a^i_{Endogenous}, a^i_{Final})\right\}$ for each element of the idiosyncratic state space. Just as in period $T-1$, we would like to have the policy functions defined over the same grid in each period, denoted in the $T-1$ problem as a_{Start}. To obtain the policy function over each point in a_{Start}, we use a combination of interpolation and grid search methods over the $\left\{(a^i_{Endogenous}, a^i_{Final})\right\}$ grids for each idiosyncratic state to obtain the optimal policy function pairs $\{(a_{Start}, a'(a_{Start}))\}$ as described in the following discussion.

If policy functions are continuous, standard algorithms can interpolate over the the obtained policy functions to obtain approximations that are only subject to interpolation errors. In our model, standard interpolation methods fail due to discontinuities in the policy functions $a'(a, y, t, s, h, g)$. A discrete jump in the policy function creates complications, as interpolation will assign an off-grid policy function value based on a continuous functional approximation of the known points on the grid. In the presence of discontinuities, it may be that the optimal policy is better approximated only using points below or above the off-grid point, depending on which side of the discrete jump it falls. These discontinuities emerge because the value function is bimodal in a', as illustrated in Figure 1 of the main paper.

The two modes represent a case where the consumer has enough money to self insure against using government care in the healthy states, but not in the LTC state. For consumers with more wealth, the higher value mode is the one associated with a higher level of savings because the consumer has enough wealth that he is better off saving to self insure against future health and LTC shocks instead of relying on government provided care. At lower levels of current wealth, the consumer is better off consuming more today and only insuring against the potential use of state 0 and 1 government care, but expecting to use government care in a future LTC state. (A similar discrete jump in the policy function can occur where the consumer only self insures against government care in state 0 and plans to use government care should they transition to state 1 or 2 next period). At the threshold where the value function jumps from the lower to the higher mode (in main paper Figure 1, at approximately $a = 68$), there will be a discrete jump in the policy function (although the value function will remain continuous).

Such jumps are a potential problem for interpolation. Suppose in main paper Figure 1 we waned to find the savings policy when $a = 68$, but we only observed the (approximate) saving policies $(64, 38)$ and $(72, 57)$. Naive interpolation would indicate a policy pair of $(68, 47.5)$. However, an agent in this state would strictly prefer $a' = 40$ or $a' = 54$ to $a' = 47.5$. This example demonstrates both the presence and the problems associated with this type discrete jump in the optimal policy function. Interpolation over regions that include jumps would lead to incorrect solutions.

The above illustrations also provide insight into why some points are only local maxima and are thus discarded in step 2 of the algorithm above. The points that we discard tend to be clumped together because of saving for lumpy costs. The local max in a value function correspond to self insuring for a relevant risk, or to forgo insurance and consume more. Thus, there exist asset levels at which the self insurance motive becomes dominant, and the local maximum associated with higher immediate consumption becomes lower than the other local maximum associated with self insurance. At this point, the optimal savings decision switches, generating the discrete jumps in the savings policy. These saving motives tend to be monotonic: if an agent with a_1 initial assets should self insure against health state s, then an agent with a_2 initial assets should also self insure against the same health state. Thus, when a local max is the highest local max, it is the highest local max for a connected region. This is the key insight that allows us to propose the following alternative to naive interpolation.

- Start from the grid $\{(a^i_{Endogenous},\ a^i_{Final})\}$ that has been restricted to globally optimal pairs. For each element of the grid indexed by k, i.e., $(a^k_{Endogenous},\ a^k_{Final}) \in \left\{(a^i_{Endogenous},\ a^i_{Final})\right\}$:
  - If $(a^{k-1}_{Endogenous},\ a^{k-1}_{Final}) \in \left\{(a^i_{Endogenous},\ a^i_{Final})\right\}$ then we interpolate for all $a^m \in a_{Start}$ in the region $a^m \in [a^{k-1}_{Endogenous},\ a^k_{Endogenous}]$. This yields $a'(a^m) \in [a^{k-1}_{Final},\ a^k_{Final}]$. Similarly, if $(a^{k+1}_{Endogenous},\ a^{k+1}_{Final}) \in \left\{(a^i_{Endogenous},\ a^i_{Final})\right\}$ then we interpolate for all $a^m \in a_{Start}$ in the region $a^m \in [a^k_{Endogenous},\ a^{k+1}_{Endogenous}]$. This yields $a'(a^m) \in [a^k_{Final}, a^{k+1}_{Final}]$.
  - If $(a^{k-1}_{Endogenous},\ a^{k-1}_{Final}) \notin \left\{(a^i_{Endogenous},\ a^i_{Final})\right\}$, then for any $a^m \in a_{Start}$ with $a^m \in [a^{k-1}_{Endogenous}, a^k_{Endogenous}]$, we do not interpolate. Instead, we run a grid search to find $a'(a^m)$. Similarly, if $(a^{k+1}_{Endogenous},\ a^{k+1}_{Final}) \notin \left\{(a^i_{Endogenous},\ a^i_{Final})\right\}$, then for any $a^m \in [a^k_{Endogenous}, a^{k+1}_{Endogenous}]$ we do not interpolate. Instead, we run a grid search to find $a'(a^m)$.

The above steps ensure there is no interpolation over a point of discontinuity. If we haven't discarded any grid points for a region on the a_{Final} grid that is connected, then the policy function is continuous for the range defined by this connected region. Thus, we are able to interpolate over the corresponding domain. If we have discarded points, then it could imply a discontinuity of the policy function in this range. Thus, we do not interpolate over the corresponding domain. As always, this is repeated for each y, s, h, g.

Finally, notice that the above fix may be foregoing interpolation in favor of a grid search method (or constrained optimization) when it doesn't necessarily need to. If a region of the domain for which a policy function is continuous is small (i.e., has few endogenous grid points), we may be ignoring a large portion of this region that could be interpolated over, and instead are executing grid searches. The grid search will return an optimal policy (up to degree of tolerance specified in the grid). Technically the entire model could be solved with grid searches on a very fine grid and yield precise solutions. Such a grid search, however, would be (prohibitively) computationally costly. Favoring a grid search method over interpolation in regions with potential discontinuities is a decision favoring a slower but more accurate algorithm in regions where complications are likely to exist.

To present a brief illustrative example, suppose we started out with the following grid $(a_{Endogenous}, a_{Final})$ for a given y, s, h and g:

$a_{Endogenous}$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_{Final}	a'_1	a'_2	a'_3	a'_4	a'_5	a'_6	a'_7	a'_8	a'_9	a'_{10}	a'_{11}	a'_{12}

and suppose that after discarding non global arg max we are left with

$a_{Endogenous}$	a_1	a_3	a_4	a_6	a_8	a_9	a_{10}	a_{11}	a_{12}
a_{Final}	a'_1	a'_3	a'_4	a'_6	a'_8	a'_9	a'_{10}	a'_{11}	a'_{12}.

If $a^i \in a_{Start}$ such that $a^i \in [a_3, a_4]$ or $a^i \in [a_8, a_{12}]$, then we interpolate over the $(a_{Endogenous,} a_{Final})$ grid. If $a^i \in [0, a_3)$, $a^i \in (a_4, a_6)$, or $a^i \in (a_6, a_8)$ then we do not interpolate, and instead perform a grid search.

Having defined the decision rule for when to use interpolation and when to use grid search methods, we can use the appropriate method starting from a_{Start} to obtain the policy function pairs $(a_{Start}, a'(a_{Start}))$. This completes the description of how to implement point 3 of the algorithm.

4. Having defined the candidate optimal policy functions, we must check that they do not violate constraints. Notice that by constructing the a_{Final} grid such that $a^i > 0 \ \forall a^i \in a_{Final}$ the $a' > 0$ constraint is satisfied by construction. For the policy function pair $(a^0_{Endogenous}, 0)$ if $a^0_{Endogenous} > 0$, then we know that $a^0_{Endogenous}$ is the point at which our non-negativity constraint binds. In the interpolation procedure detailed above, the policy function $a'(a^i) = 0$ for all $a^i < a^0_{Endogenous}$. Thus, all that needs to be checked is the LTC expenditure constraint: $c^i_{T-j} \geq \chi$ if $s = 2$. If this allocation is not budget feasible, then we know that $G = 1$ and know the agent will use government care. If this constraint is budget feasible but not satisfied by the previously calculated policy function $a'(a^i_{start})$, we run a grid search to find the optimal $a'(a^i_{start}) \in [0\ ,\ (1+r)a^i_{start} + y - h - \chi]$. It is likely that the grid search will return a value of $a'(a^i_{start}) = (1+r)a^i_{start} + y - h - \chi$ or equivalently $e_{LTC} = \chi$, but because of the non-convexities (and inability to ensure monotonicity of our savings policy function, we err on the side of caution and execute a grid search over the entire budget feasible policy set.

   This completes the computation of the optimal policy functions for the case of $G = 0$.

5. The optimal policy functions, $a'^{,i}(a^i)$ for each point $a^i \in a_{Start}$ for each element of the idiosyncratic state space, are then used to compute the value function without government care $(G = 0)$, $\hat{J}(a^i, y, T - 1, s, h, g, a'^{,i})$. This is done over the entire a_{Start} grid (for each state (y, s, h, g)). The value of using government care $(G = 1)$ is

$$GC(a^i, y, T - j, s) = U_s(\omega_G, \psi_G) + ECV_{T-j}(0, s)$$

   for each grid point. Finally, we calculate the final value function is

$$V(a^i, y, T - j, s, h, g) = max\left(\hat{J}(a^i, y, T - j, s, h, g, a'^{,i})\ ,\ GC(a^i, y, T - j, s)\right).$$

   if the the budget constraint can be satisfied, and

$$V(a^i, y, T - h, s, h) = GC(a^i, y, T - j, s)$$

if positive consumption is not budget feasible due to health-care costs.

This algorithm computes the value function value for each $s \in \{0,1,2\}$ over the fixed grid a_{Start}. The value of death when ($s = 3$ is $v(\max\{0,(1+r)a^i - h\})$. Combining these generates a grid containing the value function associated with each element of the idiosyncratic state variable grid (including assets (a)) for the $T-j$ problem. This algorithm can be repeated for $T-j$ problem incrementing j and iterating until $T-j = t_0$.

4 Deriving the FOC and ECV

In this section we derive the FOC and analytical expressions for the continuation value for $t \leq T-2$

We proceed by deriving the FOCs assuming that the consumer chooses not to use government care, i.e., $G = 0$. As is outlined above, the optimal choice of G is determined by comparing the value for $G = 0$ computed below to the value associated with $G = 1$. This problem, for $j \geq 2$, is defined by

$$J(a,y,T-j,s,h,g) = \max_{a',c}[U_s(c)] + \beta E[V(a',y,T-j+1,s',h',g)].$$

Substituting in the expected continuation value yields

$$\begin{aligned} ECV_{T-j}(a',s) &= \beta \sum_{s\in\{0,1,2,3\}} \pi(s'|s) \\ &\quad \times \int_{\Omega_{H|T-j+1,s'}} V(a',y,T-j+1,s',h',g)dH(h'|s',T-j+1) \\ &= \beta E[V(a',y,T-j+1,s',h',g)]. \end{aligned}$$

Denoting Lagrange multipliers by λ in parenthesis, the optimization problem can be expressed as

$$\begin{aligned} J(a,y,T-j,s,h,g) = \max_{c,a'} & \; U_s(c) + ECV_{T-j}(a',s) \\ & st. \\ & (1+r)a + y(T-j) - a' - c - h \geq 0 && (\lambda_{BC}^{T-j}) \\ & a' \geq 0 && (\lambda_W^{T-j}) \\ & c \geq \chi \text{ if } (G = 1 \;\&\; s = 2). && (\lambda_\chi^{T-j}) \end{aligned}$$

4.1 FOCs

The FOCs for the given states are

$s \in \{0,1\}$

FOC(c):

$$\frac{d}{dc}[U_s(c)] - \lambda_{BC}^{T-j} = 0$$

FOC(a'):

$$\frac{d}{da'}\left[ECV_{T-j}(a',s)\right] - \lambda_{BC}^{T-j} + \lambda_W^{T-j} = 0.$$

Thus, there are 4 unknowns (λ_{BC}^{T-j} λ_{W}^{T-j}, c, a') and 4 equations:

$$\begin{aligned}\frac{d}{dc}\left[U_s(c)\right] - \lambda_{BC}^{T-j} &= 0\\ \frac{d}{da'}\left[ECV_{T-j}(a',s)\right] - \lambda_{BC}^{T-j} + \lambda_{W}^{T-j} &= 0\\ \lambda_{BC}^{T-j} \times \big((1+r)a + y(T-j) - a' - c - h\big) &= 0\\ \lambda_{W}^{T-j} \times a' &= 0.\end{aligned}$$

$s = 2$

FOC(e_{LTC})

$$\frac{d}{dc}\left[U_{s=2}(c)\right] - \lambda_{BC}^{T-j} + \lambda_{\chi}^{T-j} = 0$$

FOC(a')

$$\frac{d}{da'}\left[ECV_{T-j}(a',s)\right] - \lambda_{BC}^{T-j} + \lambda_{W}^{T-j} = 0.$$

Thus, there are 5 unknowns (λ_{BC}^{T-j} λ_{W}^{T-j}, λ_{χ}, c, a') and 5 equations:

$$\begin{aligned}\frac{d}{dc}\left[U_{s=2}(c)\right] - \lambda_{BC}^{T-j} + \lambda_{\chi}^{T-j} &= 0\\ \frac{d}{da'}\left[ECV_{T-j}(a',s)\right] - \lambda_{BC}^{T-j} + \lambda_{W}^{T-j} &= 0\\ \lambda_{BC}^{T-j} \times \big((1+r)a + y(T-j) - a' - c - h\big) &= 0\\ \lambda_{W}^{T-j} \times a' &= 0\\ \lambda_{\chi}^{T-j} \times (c - \chi) &= 0.\end{aligned}$$

4.1.1 Derivative of ECV

The endogenous grid method requires inverting the following relationships to obtain a value for expenditure c which can then yield initial assets a using the budget constraint

$$\frac{d}{dc}\left[U_s(c)\right] = \frac{d}{da'}\left[ECV_{T-j}(a',s)\right].$$

The expression $\frac{d}{da'}\left[ECV_{T-j}(a',s)\right]$ can be written as

$$\begin{aligned}
\frac{d}{da'}\left[ECV_{T-j}(a',s)\right] &= \frac{d}{da'}\beta \sum_{s'\in\{0,1,2,3\}} \pi(s'|s) \\
&\quad \int_{\Omega_{H|T-j+1,s'}} V(a',y,T-j+1,s',h',g)dH(h'|s',T-j+1) \\
&= \beta \sum_{s'\in\{0,1,2,3\}} \pi(s'|s) \times \\
&\quad \int_{\Omega_{H|T-j+1,s'}} \frac{d}{da'}\left[V(a',y,T-j+1,s',h',g)\right] dH(h'|s',T-j+1).
\end{aligned}$$

Assuming $G = 0$, substitute in the optimal policy function c.

For $s' \in \{0,1\}$, by the envelope theorem,

$$\begin{aligned}
\frac{d}{da'}\left[V(a',y,T-j+1,s',h',g)\right] &= \frac{d}{da'}[U_s(c') + \beta V(a'',y,T-j+2,s'',h'',g) \\
&\quad +\lambda_{BC}^{T-j+1}((1+r)a' + y' - a'' - c' - h') \\
&\quad +\lambda_{W}^{T-j+1}(a'')] \\
&= \lambda_{BC}^{T-j+1}(1+r)
\end{aligned}$$

Plugging back in the equilibrium expression for λ_{BC}^{T-j+1} shows that

$$\frac{d}{da'}\left[V(a',y,T-j+1,s',h',g)\right] = \frac{d}{dc}\left[U_s(c')\right](1+r).$$

For $s' = 2$,

$$\begin{aligned}
\frac{d}{da'}\left[V(a',y,T-j+1,s',h',g)\right] &= \frac{d}{da'}[U_{s=2}(c') + \beta V(a'',y,T-j+2,s'',h'',g) \\
&\quad +\lambda_{BC}^{T-j+1}((1+r)a' + y' - a'' - c' - h')) \\
&\quad +\lambda_{W}^{T-j+1}(a'') \\
&\quad +\lambda_{\chi}^{T-j+1}(c' - \chi)] \\
&= \lambda_{BC}^{T-j+1}(1+r).
\end{aligned}$$

Using the envelope theorem and substituting in $\lambda_{BC}^{T-j+1} = \frac{d}{dc}\left[U_{s=2}(c')\right] + \lambda_{\chi}^{T-j+1}$ yields

$$\frac{d}{da'}\left[V(a',y,T-j+1,s',h')\right] = (1+r) \times \frac{d}{dc}\left[U_{s=2}(c')\right] + (1+r)\lambda_{\chi}^{T-j+1}.$$

Since $\lambda_\chi = 0$ if the constraint is not binding by the complementary slackness condition, the above simplifies to

$$\frac{d}{da'}\left[V(a',y,T-j+1,s',h')\right] = \begin{cases} (1+r) \times \frac{d}{dc}U_{s=2}(c') & \text{if } c > \chi \\ (1+r) \times \frac{d}{dc}U_{s=2}(c') + (1+r)\lambda_{\chi}^{T-j+1} & \text{if } c = \chi. \end{cases}$$

It remains to solve for λ_χ^{T-j+1}. Since c', a'' are known due to backwards induction, the system reduces to

$$\begin{aligned}
\frac{d}{dc}\left[U_{s=2}(c')\right] - \lambda_{BC}^{T-j+1} + \lambda_\chi^{T-j+1} &= 0 \\
\frac{d}{da'}\left[ECV_{T-j}(a'',s')\right] - \lambda_{BC}^{T-j+1} + \lambda_W^{T-j+1} &= 0 \\
\lambda_{BC}^{T-j+1} \times \big((1+r)a' + y(T-j+1) - a'' - c' - h'\big) &= 0 \\
\lambda_W^{T-j+1} \times a'' &= 0 \\
\lambda_\chi^{T-j+1} \times (c-\chi) &= 0,
\end{aligned}$$

yielding for the three unknown multipliers:

$$\begin{aligned}
-\frac{d}{de_{LTC}}\left[U_{s=2}(c')\right] + \lambda_{BC}^{T-j} &= \lambda_\chi^{T-j+1} \\
\frac{d}{da'}\left[ECV_{T-j}(a'',s')\right] + \lambda_W^{T-j} &= \lambda_{BC}^{T-j}.
\end{aligned}$$

Given that utility is strictly increasing in c or e_{LTC} for the relevant health state, the budget constraint holds with equality and $\lambda_{BC} = 0 \; \forall \; t$.
Thus,

$$\lambda_\chi = -\frac{d}{dc}\left[U_{s=2}(c')\right]$$

and

$$\frac{d}{da'}\left[V(a',y,T-j+1,s'=2,h')\right] = \begin{cases} (1+r) \times \frac{d}{dc}\left[U_{s=2}(c')\right] & \text{if } \; c > \chi \\ 0 & \text{if } \; c = \chi. \end{cases}$$

This expression provides a complete characterization of the continuation values as follows:

$$\begin{aligned}
\frac{d}{da'}\left[V(a',y,T-j+1,s'\in\{0,1\},h'))\right] &= (1+r)\frac{d}{dc}\left[U_s(c')\right] \\
\frac{d}{da'}\left[V(a',y,T-j+1,s'=2,h'))\right] &= \begin{cases} (1+r)\frac{d}{dc}\left[U_{s=2}(c')\right] & \text{if } \; c > \chi \\ 0 & \text{if } \; c = \chi \end{cases} \\
\frac{d}{da'}\left[V(a',y,T-j+1,s'=3,h'))\right] &= \begin{cases} \frac{d}{da'}[v((1+r)a'-h')] & \text{if } \; (1+r)a'-h' > 0 \\ 0 & \text{if } \; (1+r)a'-h' \leq 0. \end{cases}
\end{aligned}$$

Substituting the above three expressions into the below completely characterizes the derivative of the expected continuation value. Thus, we have derived the following Euler equations:

$s \in \{0,1\} \implies$

$$\frac{d}{dc}\left[U_s(c)\right] = \beta \sum_{s' \in \{0,1,2,3\}} \pi(s'|s) \int_{\Omega_{H|T-j+1,s'}} \frac{d}{da'}\left[V(a', y, T-j+1, s', h', g)\right] dH(h'|s', T-j+1) + \lambda_W^{T-j}$$

$s = 2 \Longrightarrow$

$$\frac{d}{dc}\left[U_{s=2}(c)\right] = \beta \sum_{s' \in \{0,1,2,3\}} \pi(s'|s) \int_{\Omega_{H|T-j+1,s'}} \frac{d}{da'}\left[V(a', y, T-j+1, s', h', g)\right] dH(h'|s', T-j+1) + \lambda_W^{T-j} - \lambda_\chi^{T-j}.$$

To handle the case in which the consumer chooses to use government care realize that for each period $T-j+1$ and for each element of the idiosyncratic state grid, there exists an $\bar{a}^{T-j+1}$ such that if $a^{T-j+1} < \bar{a}^{T-j+1}$ then $G = 1$. If $G = 1$, then

$$\frac{d}{da'}\left[V(a', y, T-j+1, , s, h, g)\right] = \frac{d}{da'}\left[\left[U_s(\omega_G, \psi_G)\right] + \beta E[V(0, y, j+1, s', h', g)]\right] = 0,$$

since a marginal change in wealth does not affect the continuation value. Thus,

$$\frac{d}{da'}\left[ECV_{T-j}(a', s)\right] = \beta \sum_{s' \in \{0,1,2,3\}} \pi(s'|s) \times \mathbb{I}_{\left\{a^{T-j+1} > \bar{a}_s^{T-j+1}\right\}} \int_{\Omega_{H|T-j+1,s'}} \frac{d}{da'}\left[V(a', y, T-j+1, s'h', g)\right] dH(h'|s', T-j+1).$$

4.1.2 Summary of FOCs

The above work derived analytic expressions for the model objects that characterize the FOCs of an agent's decision problem in period $t \leq T-2$. Using the utility function associated with health state s, the FOCs are given by

$$\frac{d}{dc}\left[U_s(c)\right] = \frac{d}{da'}\left[ECV_{T-j}(a', s)\right]$$

In the above sections we derived analytical expressions for $\frac{d}{da_{t+1}}\left[ECV_t(a_{t+1}, s)\right])$. These expressions are presented below, and may be substituted into the body of this document to complete the specification of the algorithm designed to compute optimal policies for the model in the main paper.

$s \in \{0, 1\} \Longrightarrow$

$$\frac{d}{da'}\left[ECV_{T-j}(a', s)\right] = \beta \sum_{s' \in \{0,1,2,3\}} \pi(s'|s) \times \mathbb{I}_{\{a' > \bar{a}'_s\}} \int_{\Omega_{H|T-j+1,s'}} \frac{d}{da'}\left[V(a', y, T-j+1, s', h', g)\right] dH(h'|s', T-j+1) + \lambda_W^{T-j}$$

$s = 2 \Longrightarrow$

$$\frac{d}{da'}\left[ECV_{T-j}(a', s)\right] = \beta \sum_{s' \in \{0,1,2,3\}} \pi(s'|s) \times \mathbb{I}_{\{a' > \bar{a}'_s\}} \int_{\Omega_{H|T-j+1,s'}} \frac{d}{da'}\left[V(a', y, T-j+1, s', h', g)\right] dH(h'|s', T-j+1) + \lambda_W^{T-j} - \lambda_\chi^{T-j}.$$

with the derivatives of the value function given by

$$\frac{d}{da'}\left[V(a', y, T-j+1, s' \in \{0,1\}, h', g)\right] = (1+r)\frac{d}{dc}\left[U_s(c')\right]$$

$$\frac{d}{da'}\left[V(a', y, T-j+1, s' = 2, h', g)\right] = \begin{cases} (1+r)\frac{d}{dc}\left[U_s(c')\right] & \text{if } c > \chi \\ 0 & \text{if } c = \chi \end{cases}$$

$$\frac{d}{da'}\left[V(a', y, T-j+1, s' = 3, h', g)\right] = \begin{cases} \frac{d}{da'}[v((1+r)a' - h') & \text{if } (1+r)a' - h' > 0 \\ 0 & \text{if } (1+r)a' - h' \leq 0. \end{cases}$$

To apply EGM to the above expressions, we will make the appropriate substitutions and ignore multipliers. Having solved the problem, we will then go back and check if the constraints are satisfied. If they are, then we have shown that the Lagrange multipliers are 0, as we assumed. If not, we impose the constraints, and proceed as before.

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